IOI 2024 Solutions: Problem Mosaic

Problem Information

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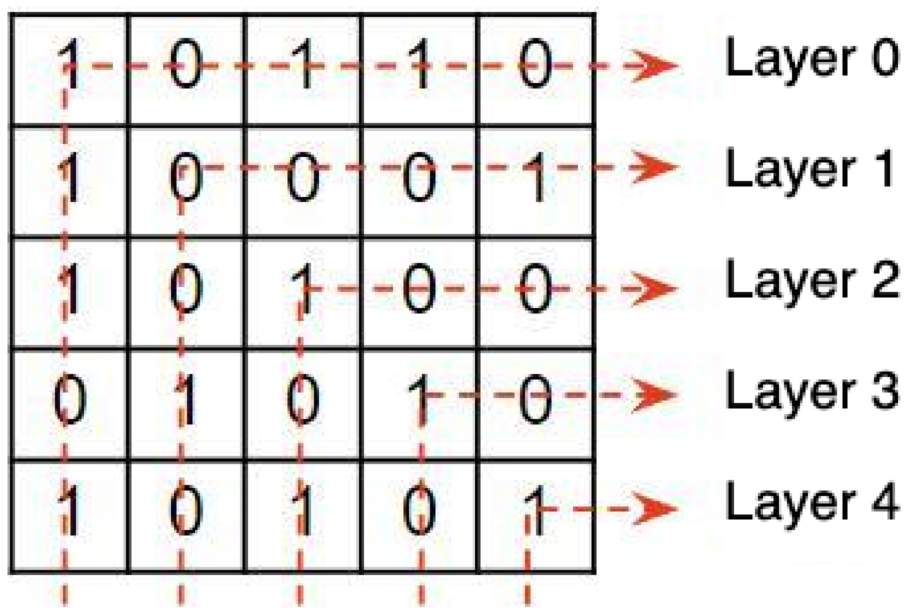
Solution

The following is the full solution. The solution for each subtask is on a section after the full solution.

Define layer `i` as all cells with row `x` and column `y` such that `min(x, y) = i`. After flattening, each layer can be regarded as a sequence of binary numbers.



Observations

The properties of each layer are as follows.

1. **Layer 0**: This is the input layer.
2. **Layer 1**: Contains no two consecutive `1`s.
3. **Layer 2**: Contains no two consecutive `1`s and no three consecutive `0`s.
4. **Layer 3 and beyond**: These layers are derived from Layer 2 by `shrinking`, meaning $A[i][j] = A[i-1][j-1]$.

The repetition of layer 3 and so on can be proven using induction.

Proof for the Main Diagonal

Case 1: $\mathbf{A}[i-1][i-1] = 1$

- If $\mathrm{A}[i-1][i-1] = 1$, it follows that both $\mathrm{A}[i-1][i]$ and $\mathrm{A}[i][i-1]$ must be `0` (as per property 3). Thus, we conclude that $\mathrm{A}[i][j] = 1$.

Case 2: $\mathbf{A}[i-1][i-1] = 0$

- If $\mathrm{A}[i-1][i-1] = 0$, then at least one of $\mathrm{A}[i-1][i]$ or $\mathrm{A}[i][i-1]$ must be `1` (as per property 3). Therefore, we conclude that $\mathrm{A}[i][j] = 0$.

Extension to Other Diagonals

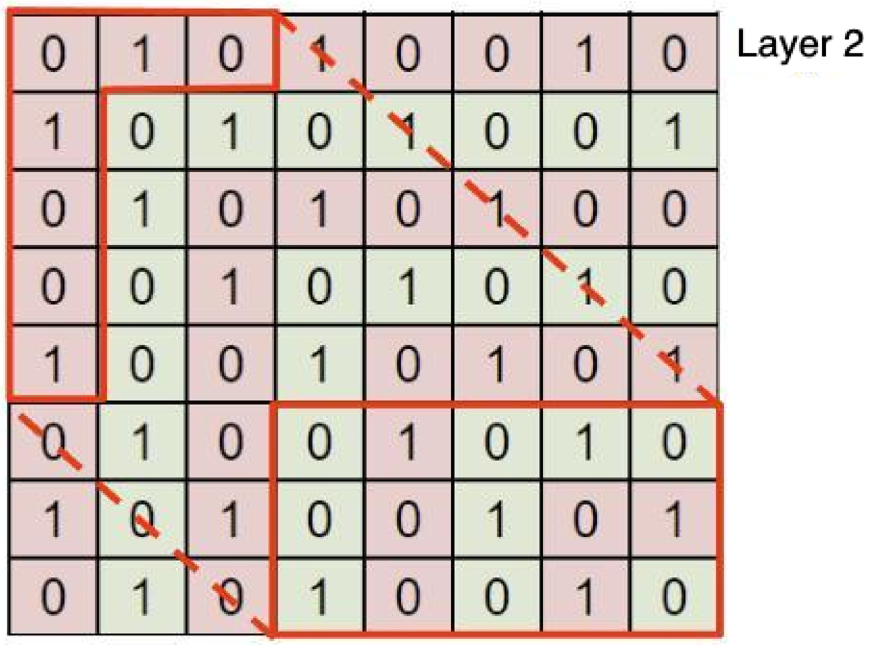
- The same reasoning can be applied to other diagonals. We can begin with the diagonals adjacent to the main diagonal and extend the proof accordingly.
- Notice that $\mathrm{A}[i][j-1]$ has the same value of the cell preceding $\mathrm{A}[i-1][j-1]$, while $\mathrm{A}[i-1][j]$ has the same value of the cell following $\mathrm{A}[i-1][j-1]$ in the layer defined by $\min(i-1, j-1)$.

From the observations above, we can start answering the queries.

To answer the query from a scenario, if the subgrid is composed of Layer 0 or Layer 1, then we will count the black cells on those layers separately (this can be done using prefix sum). Hereinafter, we assume subgrids are only composed of Layer 2 or above.

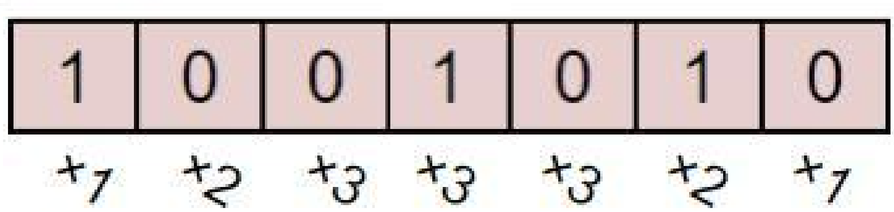
First, we find the projection of the subgrid onto Layer 2. This projection is a continuous subsequence on Layer 2.

As an example,



From the projection, we count the contribution of each element corresponding to the diagonal thickness of the original subgrid.

Notice that the contribution starts with an increasing arithmetic sequence, then it becomes a constant, and finally ends with a decreasing arithmetic sequence.



All of these can also be computed using prefix-sum.

Time complexity: $O(N + Q)$

Subtask 1 and 2

- $N, Q \leq 200$
- In this subtask, we explicitly simulate the resulting subgrid and count the number of black tiles.

Time complexity: $O(QN^2)$

Subtask 3

- $T[k] = B[k] = 0$
- In this subtask, a subgrid only occupies the first row. To solve this subtask, you need to use prefix sum to count the black tiles.

Time complexity: $O(N + Q)$

Subtask 4

- $N \leq 5000$
- In this subtask, the grid is filled explicitly. Then, for each query, the answer is calculated using a 2D prefix sum.

Time complexity: $O(N^2 + Q)$

Subtask 5

- $X[i] = Y[i] = 0$
- In this subtask, layer 0 is completely filled with zeros, while layer 1 and the subsequent layers form a checkerboard pattern. To answer a query, we first exclude layer 0 from the subgrid and calculate the area of the subgrid, dividing the result by two.
- For odd areas, be careful—the answer is given by: $\frac{\text{area} + 1}{2} - (T[k] + B[k]) \bmod 2$

Time complexity: $O(Q)$

Subtask 6

- $T[k] = B[k]$ and $L[k] = R[k]$
- In this subtask, the subgrid consists of a single cell, and its value can be determined by projecting it onto layer 2.

Time complexity: $O(N + Q)$

Subtask 7

- $T[k] = B[k]$
- This subtasks requires almost all the observations. However, the contribution of the each of the projected elements is 1, so the prefix sum that is needed is much simpler.

Time complexity: $O(N + Q)$